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Self-similar solutions of the shallow-water equations representing gravity currents with variable inflow

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A phase-plane method is used to study the existence of similarity solutions of the two-dimensional and axisymmetric shallow-water equations representing gravity currents with volumes proportional to t^{α}, where $\alpha \geqslant 0$ and t is the time after the flow is initiated. Only currents for which there is a balance between the inertia of the current and the driving buoyancy force are considered. It is found that similarity solutions exist for the two-dimensional problem for all $\alpha \geqslant 0$, with some restrictions on the condition at the current front. It is shown that no similarity solutions satisfying the boundary conditions on the axis of symmetry exist for the axisymmetric problem except when $\alpha = 0$.

1. Introduction

When a storage tank or pipeline containing a heavier-than-air gas is punctured or ruptured, its contents may form a gravity current that spreads horizontally along the ground. To assess the risks associated with the accidental release of toxic or flammable heavy gases, it is necessary to be able to estimate how rapidly this gravity current spreads. The spreading rate depends on the external flow, the surrounding terrain and the release conditions. Large-scale trials on the motion and dispersion of a cloud of heavy gas have been carried out recently at Thorney Island, UK as part of the UK Health and Safety Executive's research programme on the atmospheric dispersion of heavy gases, McQuaid (1985). Of particular interest in the present paper is how the spreading rate depends on the release conditions when the release occurs on a flat surface and in a calm environment.

Gravity currents, which are gravity-driven flows that consist of a fluid of one density moving into a fluid of a different density, are a common feature of many natural and industrial situations. Numerous examples and the basic theory of gravity currents are reviewed by Simpson (1982). Most laboratory and theoretical studies have concentrated on the spreading behaviour of gravity currents produced by the instantaneous release of a fixed volume of heavy fluid. This corresponds to a situation when the sides of a storage tank rapidly collapse releasing all of its contents at once.

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A more realistic situation is when the contents of a tank or pipeline are released over a period of *time* at a variable rate, due to, for example, a small puncture.

The spreading behaviour of gravity currents produced by the instantaneous release of a fixed volume of salt water in fresh water has been studied recently in the laboratory by Simpson & Britter (1979), Huppert & Simpson (1980) and Rottman & Simpson (1983, 1984). They found that if viscous effects remain unimportant then eventually the horizontal length of the gravity current becomes proportional to $t^{2/(n+3)}$, where $n = 0$ for plane, $n = 1$ for axisymmetric geometry and t is the time after release. This type of behaviour is exhibited by the similarity solutions of the depth-averaged shallow-water equations derived by Fannelop & Waldman (1972) and Hoult (1972) (when the motion is determined by a balance between the inertia of the gravity current and the driving buoyancy force). Grundy & Rottman (1985) show that these similarity solutions are stable to linear perturbations and that they are indeed the large-time limit of the solution of the initial-value problem. They also show that the perturbations decay as $t^{-\gamma}$, where $\gamma = \frac{1}{2}$ for plane flow and $\gamma \approx \frac{1}{5}$ for axisymmetric flow.

The large-time spreading rate of gravity currents whose volumes increase with time have been studied by a few investigators, but the results are not as complete as those for fixed-volume currents.

For the case of plane geometry, Maxworthy (1983) performed a series of experiments in which he pumped salt water at a variable rate into a parallel-sided channel filled with fresh water. The pumping was controlled so that the volume of salt water in the tank was proportional to t^{α}, where $\frac{3}{2} \leqslant \alpha \leqslant 3$. In those experiments in which viscous effects were unimportant, he observed that the length of the current eventually became proportional to $t^{\frac{1}{3}(\alpha+2)}$. This is the expected spreading behaviour based on dimensional analysis when it is assumed that the motion of the current is self-similar. These results imply that a similarity solution of the shallow-water equations may exist for this problem, but to our knowledge nobody has attempted to determine such a solution when $\alpha > 0$ (except for the trivial solution when $\alpha = 1$ where Britter (1979) noted that the speed and depth of the current are independent of the longitudinal coordinate).

For the case of axisymmetric geometry, Britter (1979) performed laboratory experiments in which salt water was pumped at a constant rate ($\alpha = 1$) into a sector-shaped tank filled with fresh water. He reported that when viscous effects were unimportant the radius of the current was proportional to $t^{\frac{3}{4}}$, which is the expected spreading rate when the current motion is assumed self similar. Britter also attempted to determine the similarity solutions to the shallow-water equations for this problem, but he was unable to find an analytic solution and suggested that numerical techniques may have to be used. Chen (1980) attempted to determine the similarity solution numerically and discovered a singularity. He argued that the appearance of this singularity implies that the physical flow will take the form of a succession of expanding concentric rings with the radius of each ring proportional to $t^{\frac{3}{4}}$, although his reasons for drawing this conclusion are not particularly clear. Garvine (1984) solved an initial-value problem for this flow numerically. His computed solutions show a single ring structure forming at the outer edge of the current whose radius is proportional to $t^{0.92}$. In addition, Garvine presents physically plausible arguments for modifying his numerical procedure such that a succession of rings are produced. R. E. Britter (private communication) observed multiple rings in his sector-shaped tank experiments. All these results leave us in a bit of a dilemma. Chen's singular similarity solution does not appear to be the large-time limit of the initial-value

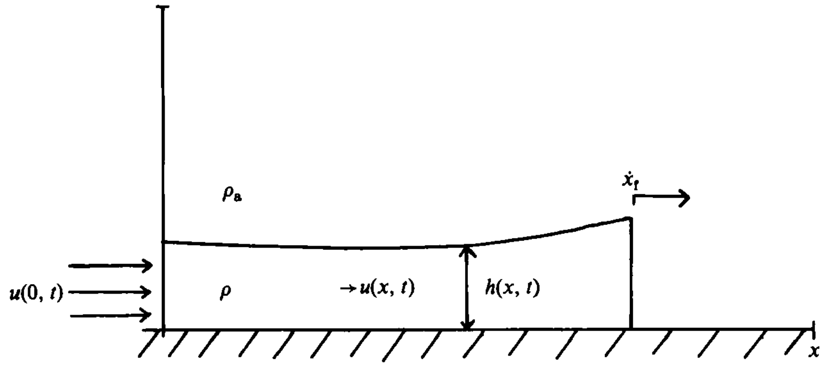

FIGURE 1. Schematic illustration of a heavy fluid with density ρ spreading along a horizontal surface through a lighter fluid of density ρ_a.

problem, since his spreading rate does not agree with Garvine's numerical results, but it nevertheless seems to agree with Britter's experimental results.

More recently, Ivey & Blake (1985) have reinterpreted Britter's data to account for some of the details of the experimental arrangement. Their replotted points indicate that the radius of the current is initially proportional to $t^{0.98}$ and later on (presumably when viscous effects become important) proportional to $t^{0.67}$. The implication is that Britter's data does not support the self-similar solution of the shallow-water equations, although they caution that more careful experiments need to be done before any definite conclusions can be drawn. It is intriguing that this reinterpretation of Britter's experimental measurements gives results for the radius of the current when viscous effects are unimportant that are quite close to those of Garvine's numerical calculations.

In the present paper we study more closely the existence and uniqueness of similarity solutions of the depth-averaged shallow-water equations for the variable inflow problem where the volume of the current is proportional to t^{α} (with $\alpha \geqslant 0$). We restrict our investigation to those flows for which there is a balance between the inertia of the current and the buoyancy force driving the current. Our approach is to use the phase-plane method developed in the context of gasdynamics by Guderley (1942), Courant & Friedrichs (1948) and Sedov (1959). We find that under certain conditions unique similarity solutions exist for the plane flow case for all $\alpha \geqslant 0$, but that no similarity solutions exist in the axisymmetric case that satisfy the boundary condition at the axis of symmetry, except when $\alpha = 0$. We also investigate the singular similarity solution found by Chen (1980) and show that the singular behaviour is spurious.

The problem is formulated in §2. The phase-plane analysis for the similarity solutions is developed in §3: the plane-flow ($n = 0$) case in §3.1 and the axisymmetric-flow ($n = 1$) case in §3.2. In §4 we summarize the results and discuss their physical implications and relation to previous work.

2. The equations and boundary conditions

We consider the motion, in both plane and axisymmetric geometries, of a current of density ρ intruding into a fluid of slightly lower density ρ_a (figure 1). The fluid in the current is introduced at the origin at some prescribed rate and flows away along a horizontal boundary, driven by the buoyancy force due to gravity and the difference in density between the two fluids. The fluids are taken to be incompressible, the depth of the lighter fluid is taken to be much greater than the thickness of the current, and any mixing between the two fluids is ignored.

We are concerned primarily with the behaviour of the current at large times after initiation, so we assume that the current's length (or radius, in axisymmetric flow) is much greater than its thickness. Then, if viscous effects are unimportant, the motion of the current can be modelled by the shallow-water equations, which in the Boussinesq approximation are

$$\frac{\partial h}{\partial t}+u\frac{\partial h}{\partial x}+h\frac{\partial u}{\partial x}+n\frac{uh}{x}=0, \tag{2.1}$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+g'\frac{\partial h}{\partial x}=0, \tag{2.2}$$

Where $h(x,t)$ is the thickness of the current, $u(x,t)$ is the depth-averaged horizontal fluid speed in the current and $g'=g(\rho-\rho_a)/\rho_a$ is the reduced acceleration due to gravity. The independent variable x represents the horizontal coordinate in plane flow and the radial coordinate in axisymmetric flow, and t measures the time after the flow is initiated. The parameter n is 0 for plane flows and 1 for axisymmetric flows. For the interested reader a derivation of these equations is given, for example, in Penney & Thornhill (1952).

We impose the boundary conditions

$$\lim_{x\to 0}\,[(2\pi x)^n uh]=\frac{\mathrm{d}}{\mathrm{d}t}(q_\alpha t^\alpha), \tag{2.3}$$

$$u(x_f,t)=\dot{x}_f(t), \tag{2.4}$$

$$\beta^2 g' h(x_f,t)=\{\dot{x}_f(t)\}^2, \tag{2.5}$$

where $x_f(t)$ denotes the position of the current front, $\dot{x}_f(t)$ its speed and β is an empirically determined constant.

The given parameters $\alpha\,(\alpha\geqslant 0)$ and $q_\alpha\,(q_\alpha>0)$ and boundary conditions (2.3) specify the volume input to the flow at the origin, and together with (2.4) determine the volume Q of the current as a function of time. To see this we integrate (2.1) with respect to x over $[0,x_f(t)]$. Using (2.4) we find that

$$\frac{\mathrm{d}Q}{\mathrm{d}t}\equiv\frac{\mathrm{d}}{\mathrm{d}t}\int_0^{x_f(t)}(2\pi x)^n h(x,t)\,\mathrm{d}x=\lim_{x\to 0}\,[(2\pi x)^n uh]. \tag{2.6}$$

Hence (2.3) gives, to within an arbitrary constant,

$$Q(t)=q_\alpha t^\alpha. \tag{2.7}$$

Note that in plane geometry Q is more precisely the volume per unit width and therefore that q_α has dimensions that depend on n and α. The requirement $\alpha\geqslant 0$ ensures that the current volume does not decrease with time, but the volume flux at the origin decreases or increases according to whether $\alpha \lesseqgtr 1$.

Boundary condition (2.5) implies a quasi-steady balance between the buoyancy force driving the current front and the drag due to the acceleration of the surrounding fluid around the front. Physical reasoning and experimental measurements suggest that the parmeter β has a value near unity for gravity currents with small values of $(\rho-\rho_a)/\rho_a$. This type of boundary condition has been used by most previous modellers of gravity currents.

In this paper we are primarily concerned with similarity solutions of the governing equations and their possible role as a model for the experimental situation. Bearing in mind our experience with the case $\alpha = 0$, where we showed that the similarity solution is the large-time limit of the solution of the initial-value problem, we expect such solutions to be similarly valid for $\alpha > 0$.

3. The similarity solutions

Dimensional analysis shows that the similarity solution of (2.1)–(2.5) has the form,

$$g'h(x,t) = (g'q_\alpha)^{2/(3+n)}\,t^{-2(1-\delta)}H(\zeta) = \delta^2\left(\frac{x}{t}\right)^2 Z(\zeta), \tag{3.1}$$

$$u(x,t) = (g'q_\alpha)^{1/(3+n)}\,t^{-(1-\delta)}U(\zeta) = \delta\left(\frac{x}{t}\right) V(\zeta), \tag{3.2}$$

$$x_f(t) = (g'q_\alpha)^{1/(3+n)}\,t^{\delta}\zeta_f, \tag{3.3}$$

where

$$\zeta = (g'q_\alpha)^{-1/(3+n)}\,xt^{-\delta} \tag{3.4}$$

is the similarity variable and

$$\delta = \frac{2+\alpha}{3+n}. \tag{3.5}$$

The parameter ζ_f is the value of ζ at $x = x_f(t)$. We have defined two sets of dimensionless functions H, U and Z, V which are related via

$$H(\zeta) = \delta^2\zeta^2 Z(\zeta), \tag{3.6}$$

and

$$U(\zeta) = \delta\zeta V(\zeta). \tag{3.7}$$

Although the functions H and U have the advantage that their profile shapes are similar to those of h and u, the functions Z and V are more convenient from a mathematical viewpoint.

Substituting (3.1)–(3.4) into (2.1) and (2.2) we find that the functions $Z(\zeta)$ and $V(\zeta)$ satisfy

$$\frac{\mathrm{d}Z}{\mathrm{d}V} = \frac{Z\{(1-V)\,[2(V-\mu)+(n+1)\,V]+V(V-\mu)+2Z\}}{\{V(V-\mu)\,(1-V)+Z[(n+1)\,V-2(1-\mu)]\}} \tag{3.8}$$

and

$$\frac{1}{\zeta}\frac{\mathrm{d}\zeta}{\mathrm{d}V} = \frac{\{(1-V)^2-Z\}}{\{V(V-\mu)\,(1-V)+Z[(n+1)\,V+2(1-\mu)]\}}, \tag{3.9}$$

where $\mu = 1/\delta$. The boundary conditions (2.3)–(2.5) become respectively

$$\lim_{\zeta\to 0}\{(2\pi\zeta)^n\,\delta^3\zeta^3 VZ\} = \alpha, \tag{3.10}$$

$$V(\zeta_f) = 1, \tag{3.11}$$

$$\beta^2 Z(\zeta_f) = 1. \tag{3.12}$$

The volume-integral condition (2.6) becomes

$$\int_0^{\zeta_f} (2\pi\zeta)^n\,\delta^2\zeta^2 Z(\zeta)\,\mathrm{d}\zeta = 1, \tag{3.13}$$

and the requirement $\alpha \geqslant 0$ implies

$$0 < \mu \leqslant \tfrac{1}{2}(3+n). \tag{3.14}$$

Although (3.13) is not an independent condition, it is useful to have it in this form.

Equation (3.9) is an autonomous first-order equation for $Z(\zeta)$ as a function of $V(\zeta)$ which has been studied extensively in the context of gasdynamics. Guderley (1942), Courant & Friedrichs (1948) and Sedov (1959), among many others, developed a powerful phase-plane method for studying the properties of the solutions of these types of equation. It is this method that we will use to investigate the existence and behaviour of the similarity solutions to our problem, the novel features of which are the boundary conditions (3.10)–(3.12).

At this point it is convenient to study the two cases $n = 0$ and $n = 1$ separately.

3.1. *Plane flow* ($n = 0$)

It turns out that the boundary condition at $\zeta = 0$ plays an important role in our analysis. Since this point corresponds to the point $V = \infty$, $Z = \infty$† in the (Z, V)-plane it is useful to make the transformation

$$V_1 = \frac{1}{V}, \tag{3.15}$$

$$W = \frac{1}{ZV_1^2}, \tag{3.16}$$

which maps this point onto the line $V_1 = 0$ in the (W, V)-plane. In terms of W and V_1, (3.8) and (3.9) become, with $\eta = \zeta/\zeta_f$,

$$\frac{dW}{dV_1} = \frac{W\{(W-4)(1-\mu)\}}{\{W(V_1-1)(1-\mu V_1)+1+2(1-\mu)V_1\}} \tag{3.17}$$

and

$$\frac{1}{\eta}\frac{d\eta}{dV_1} = \frac{\{1-W(V_1-1)^2\}}{V_1\{W(1-\mu V_1)(V_1-1)+1+2(1-\mu)V_1\}}. \tag{3.18}$$

For completeness, the boundary conditions (3.8)–(3.10) become

$$\lim_{\eta\to 0}\left\{\frac{\delta^3\zeta_f^3\eta^3}{V_1^3 W}\right\} = \alpha, \tag{3.19}$$

$$V_1(1) = 1, \tag{3.20}$$

$$W(1) = \beta^2, \tag{3.21}$$

and the volume integral condition becomes

$$\int_0^1 \delta^2\zeta_f^2\left(\frac{\eta^2}{WV_1^2}\right)d\eta = 1. \tag{3.22}$$

The phase plane of (3.17) has two basic topologies depending on whether $0 < \mu < 1$ (figure 2) or $1 < \mu < \frac{3}{2}$ (figure 3). The only singular point of relevance is marked A in both cases and is located at $V_1 = \frac{1}{2}$, $W = 4$. The front boundary condition (3.20) and (3.21) is the point $V_1 = 1$, $W = \beta^2$. The other boundary condition (3.19) is applied at $\eta = 0$ on the line $V_1 = 0$ where it can be shown that $V_1 \sim K\eta (K \neq 0)$ as $\eta \to 0$ along an integral curve cutting the W-axis. The arrows in the figures indicate the direction

† This is only true for $\alpha \neq 0$. For $\alpha = 0$ the corresponding point is $V = 1, Z = \infty$ (see Grundy & Rottman 1985).

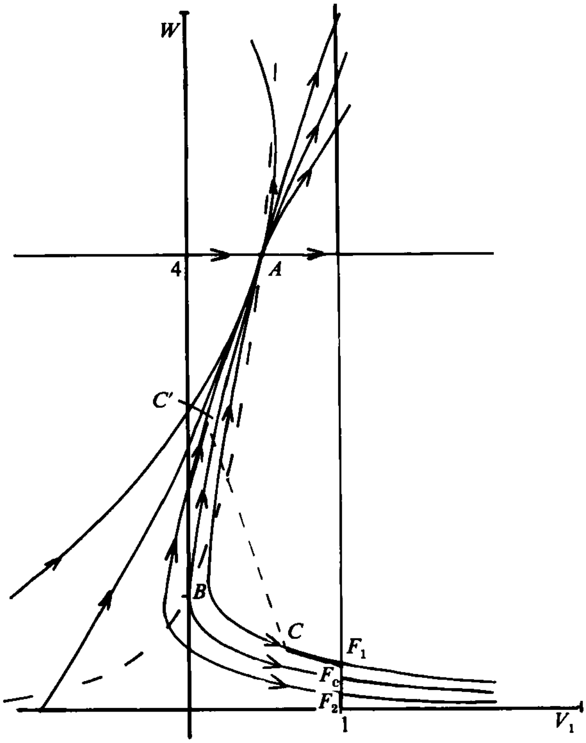

FIGURE 2. The (V_1, W)-phase plane for plane flow ($n = 0$) for $1 < \alpha < \infty$, $(0 < \mu < 1)$. The curve $F_1CC'B$ is a possible discontinuous solution, with a hydraulic jump from C to C' and $\beta \approx 1$ at F_1.

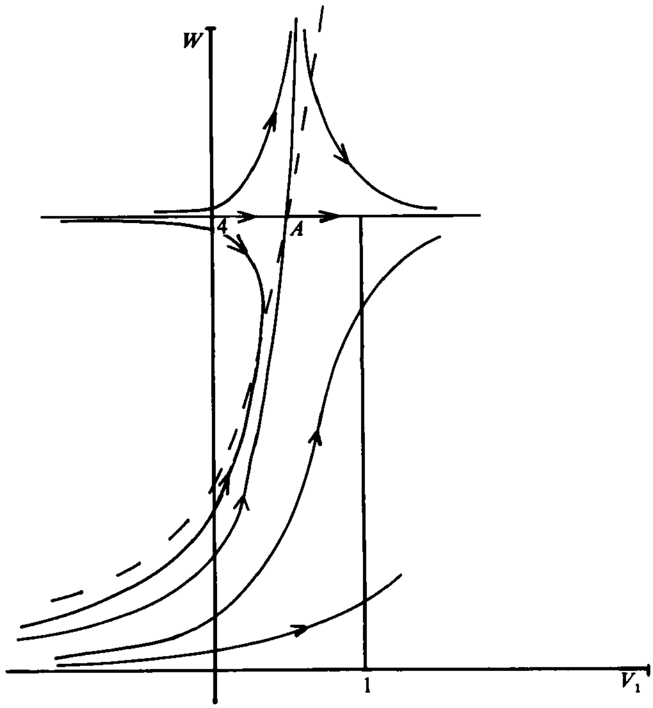

FIGURE 3. The (V_1, W)-phase plane for plane flow ($n = 0$) for $0 < \alpha < 1$ $(1 < \mu < \frac{3}{2})$.

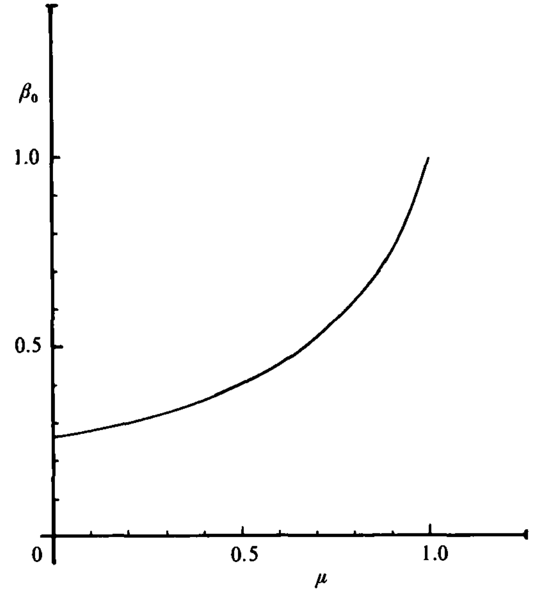

FIGURE 4. The critical value of β as a function of μ.

of increasing η and the dotted line through A is the so-called critical line, $W = 1/(V_1-1)^2$, across which η changes direction.

Let us first look at the case $0 < \mu < 1$ (figure 2), corresponding to $1 < \alpha < \infty$. Consider a point $F(V_1 = 1, W = \beta^2)$ representing the current front. To be a solution of our problem, the integral curve from F has to reach $V_1 = 0$ (where $\eta = 0$). From the figure it is clear that if $F = F_2$ is below $F_c(V_1 = 1, W = \beta_0^2)$ then $\beta < \beta_0$, and the integral curve exists and is unique. For $F = F_1$ above F_c but below the line $W = 4$, then $\beta > \beta_0$, and no integral curve from F can reach $V_1 = 0$, and so no solution exists. Lastly, if F is above $W = 4$, every integral curve from F passes through the singular point A $(\frac{1}{2}, 4)$, which is a node, and we then have an infinite choice of curves from A reaching $V_1 = 0$.

In summary, when $0 < \beta \leqslant \beta_0$ a similarity solution exists and is unique, when $\beta_0 < \beta < 2$ there is no similarity solution, and when $\beta > 2$ a similarity solution exists but is not unique. The critical value of $\beta_0(\mu)$ can be found numerically by integrating (3.17) from the point $B(0,1)$ to $V_1 = 1$ where $W = \beta_0^2$. A plot of β_0 obtained in this way is shown in figure 4; interestingly, β_0 is always less than one.

We now turn to the case $1 < \mu < \frac{3}{2}$ (figure 3), which corresponds to $0 < \alpha < 1$. The singular point A is now a saddle point and it is clear that if the point P representing the current front is below $W = 4$ in the first quadrant, then a single integral curve through P will always reach $V_1 = 0$. On the other hand if P lies above $W = 4$, then no integral curve can possibly reach $V_1 = 0$. So we conclude that, for $1 < \mu < \frac{1}{2}$, we have a unique similarity solution if $\beta < 2$ and no solution for $\beta \geqslant 2$. Grundy & Rottman (1985) showed that this is also true when $\alpha = 0$.

Finally, if $\mu = 1 (\alpha = 1)$ then all solutions collapse onto $W = \beta^2$. This gives the trivial result

$$h(x,t) = h_0 = \beta^{-\frac{2}{3}}(q_\alpha^2/g')^{\frac{1}{3}}, \tag{3.23}$$

$$u(x,t) = u_0 = \beta^{\frac{2}{3}}(g'q_\alpha)^{\frac{1}{3}}, \tag{3.24}$$

$$x_f(t) = u_0 t, \tag{3.25}$$

which is the solution noted by Britter (1979).

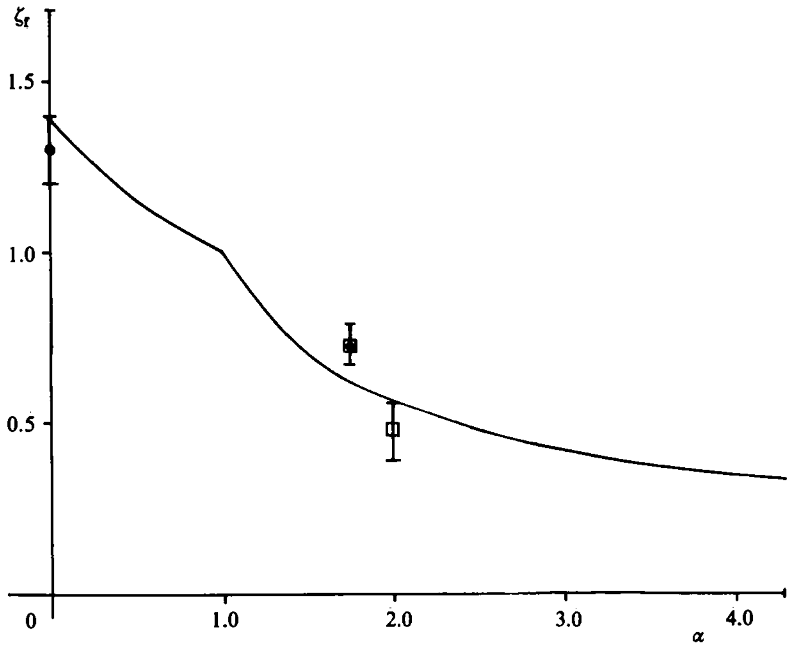

FIGURE 5. ζ_f as a function of α. Here $\beta = \beta_0(\alpha)$ for $\alpha \geqslant 1$ and $\beta = 1.0$ for $0 < \alpha \leqslant 1$. The symbols are experimental measurements (the vertical bars indicate the scatter in these measurements): ●, Huppert & Simpson (1980) and J. E. Simpson (private communication); □, Maxworthy (1983) and T. Maxworthy (private communication).

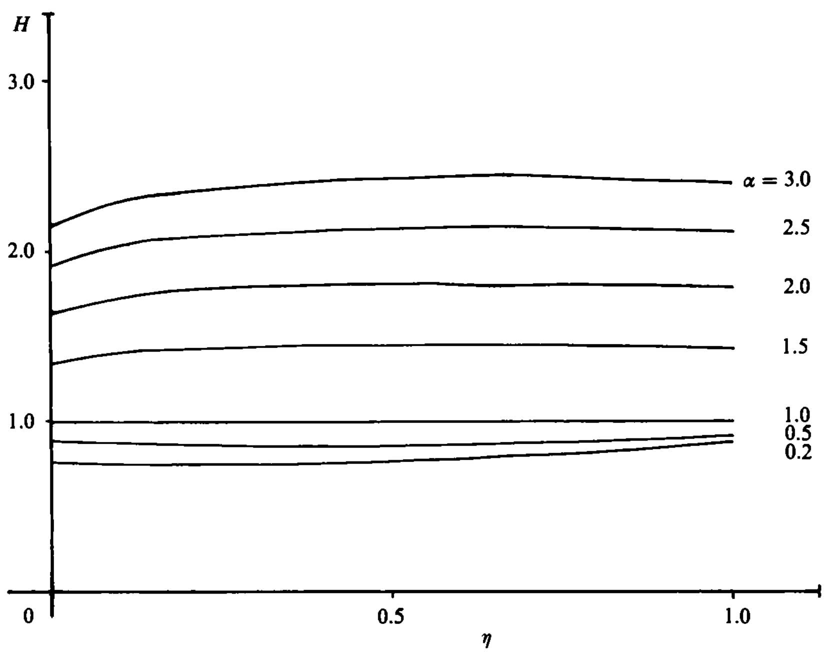

FIGURE 6. The scaled fluid depth $H(\eta)$ for various values of α. $\beta = 1.0$ for $0 < \alpha \leqslant 1$ and $\beta = \beta_0(\alpha)$ for $\alpha \geqslant 1$.

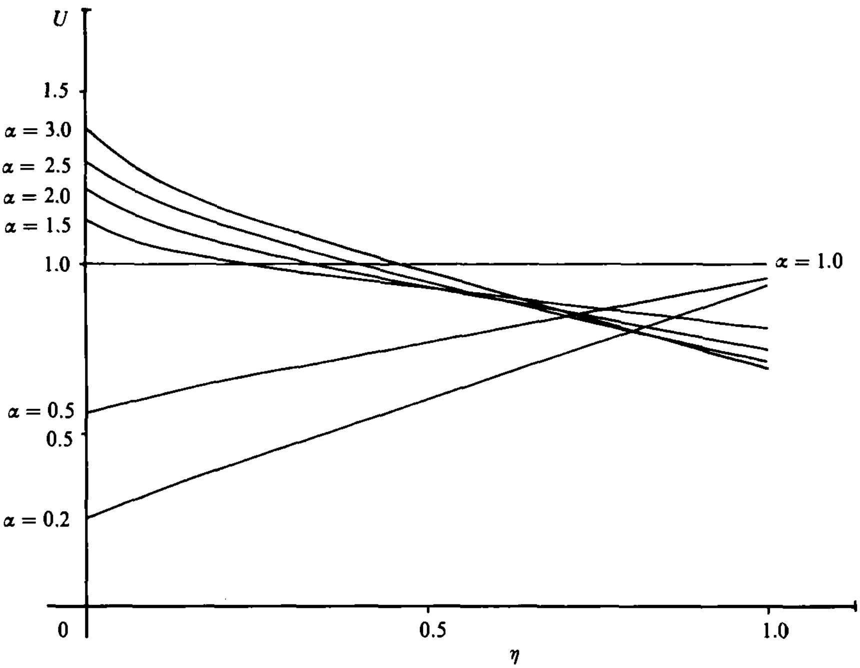

FIGURE 7. The scaled fluid speed $U(\eta)$ corresponding to the values of α and β in figure 6.

The value of ζ_f is most easily found by recasting (3.17) and (3.18) with $W(\eta)$ and $U_1(\eta) = V_1/\eta$ as the dependent variables. Starting at the current front where $W(1) = \beta^2$ and $U_1(1) = 1$, we numerically integrate this equation to obtain $W(0)$ and $U_1(0)$. It then follows from (3.19) that

$$\zeta_f = \frac{U_1(0)\,[\alpha W(0)]^{\frac{1}{3}}}{\delta}, \tag{3.26}$$

which is a function of α and β. A plot of ζ_f as a function of α is shown in figure 5.

Sample solutions for the non-dimensional fluid depth $H(\zeta)$ are plotted in figure 6 for various values of α in the domain $0 < \alpha \leqslant 3$. For $0 < \alpha < 1$ we have set $\beta = 1.0$ and for $\alpha > 1$ we have used $\beta = \beta_0(\alpha)$. The corresponding solutions for the non-dimensional fluid speed $U(\zeta)$ are plotted in figure 7. Note that $H(\zeta)$ is concave for $0 < \alpha < 1$ and convex for $\alpha > 1$ and that $U(\zeta)$ increases or decreases monotonically with ζ according to whether $\alpha \lessgtr 1$.

As we stated in the Introduction, the available experimental evidence is consistent with the conclusion of dimensional analysis, as given by (3.3), that the current length eventually becomes proportional to $t^{\frac{1}{3}(\alpha+2)}$. The experiments of Huppert & Simpson (1980) and Maxworthy (1983) also include measurements of the constant of proportionality ζ_f. We have plotted these measurements (for those cases in which viscous effects are unimportant) in figure 5 along with the values we obtained for this parameter from our similarity solutions. For this comparison, we have assumed in our similarity solutions that the empirical parameter $\beta = 1$ when $\alpha < 1$ and $\beta = \beta_0(\alpha)$ when $\alpha > 1$.

Although the agreement is quite reasonable, the experimental measurements cover a rather limited range of the parameters. Also, it is somewhat troubling that our similarity solutions do not allow us to choose arbitrarily a value of β near unity when

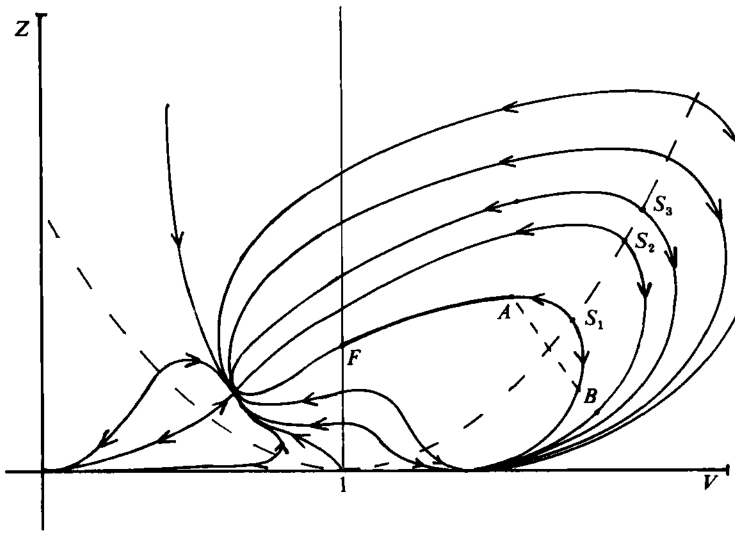

FIGURE 8. The (V, Z)-phase plane for axisymmetric flow ($n = 1$) with $\alpha = 1$.

$\alpha > 1$. From physical reasoning and experimental observations we expect $\beta \approx 1$. It may be possible, although we have not investigated this possibility in any detail, to obtain solutions for arbitrary $\beta \approx 1$ if we allow them to be discontinuous. Such a solution is illustrated in figure 2. Starting at the front position F_1 (for which $\beta > \beta_0(\alpha)$), we move backwards in η along the integral curve to a point marked C, say. Then the solution jumps to a point marked C' on another integral curve. The points C and C' are chosen such that η has the same value at both points and the changes in the dependent variables satisfy some specified hydraulic-jump conditions, such as those proposed by Garvine (1984). The solution now proceeds backwards in η along the new integral curve to the boundary $V_1 = 0$.

Finally, we remark that the solutions (3.1)–(3.4) are only consistent with our assumptions about the physical problem when $0 \leqslant \alpha < 4$. When $\alpha > 4$ these expressions indicate that the ratio of the current's thickness to its length increases with time, violating our assumptions in using the shallow-water equations to describe the motion. In addition, (3.3) gives the acceleration of the front increasing with time when $\alpha > 4$, which violates our assumption of quasi-steadiness in imposing the front boundary condition (2.5).

3.2. *Axisymmetric flow* ($n = 1$)

The situation for axisymmetric flow is somewhat different to that for plane flow. To see why this is we first of all look at the (Z, V)-phase plane for $n = 1$ which is typically shown in figure 8, the region of interest being the first quadrant with $V \geqslant 1$, $V = 1$ representing the current front. The important singular point is at $V = \infty, Z = \infty$ where $\eta = 0$. Near this point for $n = 1$ and $\alpha > 0$, with

$$Z_1 = \frac{1}{Z} \quad \text{and} \quad V_1 = \frac{1}{V},$$

(3.8) may be approximated by

$$\frac{\mathrm{d}Z_1}{\mathrm{d}V_1} = \frac{Z_1(2V_1^2 - 3Z_1)}{V_1(2V_1^2 - Z_1)}, \tag{3.27}$$

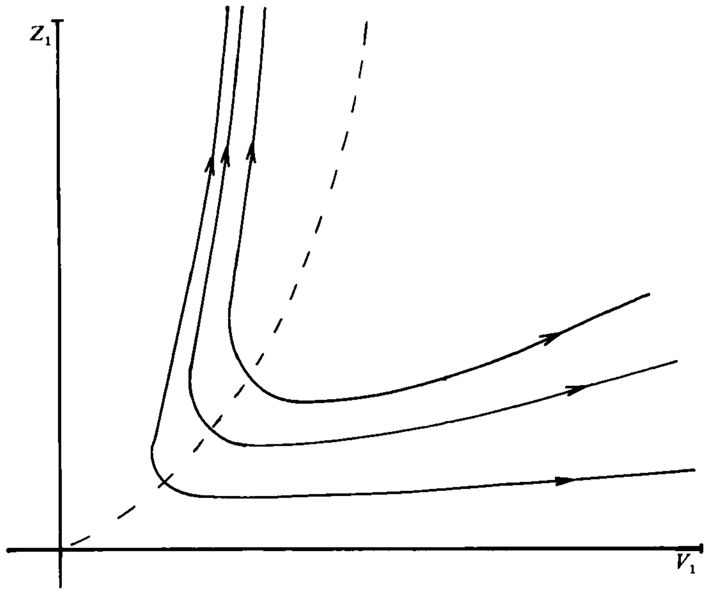

FIGURE 9. Behaviour of the integral curves near $V = \infty$, $Z = \infty$ for the axisymmetric case $n = 1, \alpha > 0$.

which can be integrated to give

$$Z_1 V_1^3 = K(2V_1^2 + Z_1)^2, \tag{3.28}$$

with an arbitrary constant K, positive for curves in the first quadrant. The explicit one parameter family of curves represented by (3.28) is shown in figure 9. We conclude from this figure that no integral curve reaches $V = Z = \infty$ from any point in the first quadrant, in particular from the line $V = 1, Z > 0$. Thus for axisymmetric flow with $\alpha > 0$ no similarity solution to our problem exists in $0 \leqslant \eta \leqslant 1$ that satisfies the boundary condition on the axis of symmetry. As shown in a previous paper, Grundy & Rottman (1985), a unique similarity solution does exist for $\alpha = 0$ since in that event η is zero at the singular point $V = 1, Z = \infty$ which is accessible from any point on $V = 1$.

We now consider the numerical solution that Chen (1980) obtained for the case with $\alpha = 1$. He solved the equations for H and U by starting at the front ($\eta = 1$) and numerically integrating backwards in η. The solution he obtained for H is shown schematically in figure 10 as a dashed line. His solution has $H \to 0$ and $U \to \infty$ as $\eta \to \eta_0$ for some $\eta_0 > 0$. Clearly this solution cannot satisfy the boundary condition at $\eta = 0$ and furthermore does not imply the correct singular behaviour of the solution. We can show this by examining the phase plane in figure 8. If we follow an integral curve from the front (marked F) we see that this must meet the critical line at some point S_1, say, where $\eta = \eta_s$, $Z = Z_s$, and $V = V_s$ with $Z_s = (1-V_s)^2$. Clearly $H(\eta_s)$ and $U(\eta_s)$ are finite but from (3.8), (3.9) and (3.6), (3.7) the derivatives $(\mathrm{d}H/\mathrm{d}\eta)_s$ and $(\mathrm{d}U/\mathrm{d}\eta)_s$ are unbounded. An examination of the equations reveals a square-root singularity with $H(\eta)$ and $U(\eta)$ having the expansions

$$H(\eta) = H(\eta_s) + O\{(\eta-\eta_s)^{\frac{1}{2}}\}, \tag{3.29}$$

$$U(\eta) = U(\eta_s) + O\{(\eta-\eta_s)^{\frac{1}{2}}\}. \tag{3.30}$$

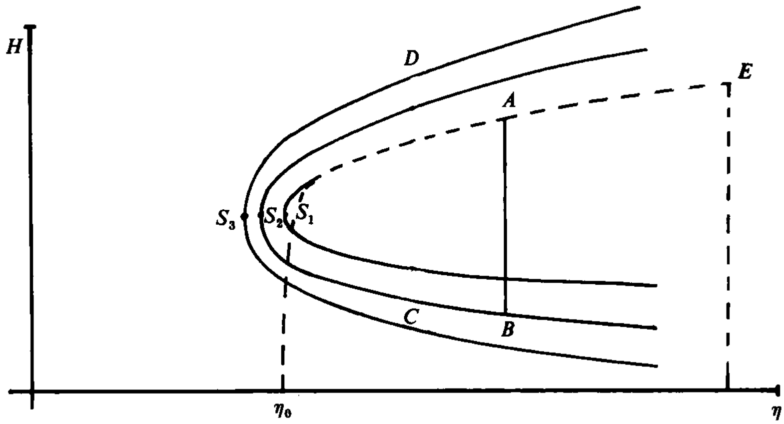

FIGURE 10. Schematic representation of the fluid depth $H(\eta)$ for axisymmetric flow ($n = 1$) with $\alpha = 1$. The dashed line is Chen's numerical solution and the solid lines represent the actual solutions, corresponding to the integral curves in figure 8.

The solutions for H corresponding to the integral curves in the phase plane are shown schematically by the solid lines in figure 10. Both $H(\eta)$ and $U(\eta)$ are double-valued functions of η for $\eta > \eta_s$ with infinite slope at $\eta = \eta_s$. Chen's numerical method appears unable to handle the infinite slope at $\eta = \eta_s$, and produces a spurious singularity near this point.

As in the case of plane geometry, we can construct discontinuous solutions for the axisymmetric case, although it should be kept in mind that these solutions cannot satisfy the boundary condition at the axis of symmetry. Such a solution is shown schematically in figure 8, where a hydraulic jump connects the point marked A to the point marked B, and the corresponding solution for H is shown schematically in figure 10. Finally, the form of the similarity solutions (3.1)–(3.4) shows that they are inconsistent with our assumptions about the physical problem when $\alpha > 6$, for the same reasons as in the case of plane geometry.

4. Summary and discussion of results

For plane flow we have shown that continuous similarity solutions of our model problem exist and are unique if the front parameter β is suitably restricted. Specifically, $0 \leqslant \beta \leqslant 2$ when $0 \leqslant \alpha < 1$ and $0 \leqslant \beta \leqslant \beta_0(\alpha)$ when $\alpha > 1$. From physical reasoning, we expect β to have a value near unity. Therefore, the non-unique solutions we obtained for $\alpha > 1$ are inappropriate in the physical situation since they exist only when $\beta > 2$. A similar comment may be relevant to some of the unique solutions we obtained for $\alpha > 1$ with $0 \leqslant \beta \leqslant \beta_0(\alpha)$, since in general $\beta_0(\alpha) < 1$ (as shown in figure 4), although the small number of experimental results that are available seem to indicate that $\beta \approx \beta_0(\alpha)$ for α not too much greater than one. Another possibility is the existence of discontinuous solutions that satisfy the front condition for a prescribed value of β. The similarity solutions of the shallow-water equations are only appropriate for the physical problem when $0 \leqslant \alpha < 4$.

For axisymmetric flow we have shown that no similarity solution of our model equations satisfying the boundary conditions on the axis of symmetry exists for $\alpha > 0$, whatever the value of β. The accumulated theoretical and experimental results seem to indicate that when $\alpha > 0$ the source conditions remain important no matter how large the spreading pool of heavy fluid becomes. This excludes the possibility of a similarity solution. However, much more work needs to be done before any definite conclusions can be made.

One final point should be mentioned concerning the volume input function $Q(t)$. Since we are primarily concerned with the large-time behaviour then we would expect the similarity solutions to be a valid approximation in this limit for volume input functions,

$$Q(t) \sim q_\alpha t^\alpha \quad \text{as } t \to \infty.$$

The results of the present paper are appropriate only when the motion is determined by a balance between the inertia of the heavy fluid and the driving buoyancy force. Huppert (1982) has shown from order-of-magnitude arguments that (for large enough time) the viscous drag due to the rigid horizontal boundary becomes more important than inertia when $\alpha < \alpha_c = \frac{1}{4}(7+5n)$. He has also shown the (at first thought, surprising) result that when $\alpha > \alpha_c$ viscous drag is more important than inertia for small time but inertia eventually dominates viscous drag for large time. When $\alpha = \alpha_c$ inertia is always dominant if $J \ll 1$ and viscous drag is always dominant if $J \gg 1$, where

$$J = (\nu^{(3+n)} g'^{2(1+n)}/q^4_{\alpha c})^{(3-2n)/(3+n)}$$

is a dimensionless number. Huppert (1982) also estimated the time,

$$t_* \sim (q^4_\alpha / g'^{2(1+n)} \nu^{(3+n)})^{1/(7+5n-4\alpha)},$$

when inertia and viscous drag are comparable. We expect our results to be valid for times large compared with a characteristic timescale based on the source conditions† but small compared with t_* if $\alpha < \alpha_c$ or large compared with t_* if $\alpha > \alpha_c$.

The self-similar solutions for the case when the motion is determined by a balance between viscous drag and the driving buoyancy force have been determined by Huppert (1982) for all $\alpha \geq 0$ and for both plane and axisymmetric flow. It is interesting that in contrast to the present results the viscous–buoyancy problem admits similarity solutions for the full range of the parameters. Experiments on the viscous–buoyancy problem have been reported by Didden & Maxworthy (1982), Huppert (1982) and Maxworthy (1983), and their results compare fairly well with the similarity solutions.

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† If the source had a finite length on radius x_0, for example, then a characteristic timescale based on the source conditions would be $t_0 \sim (x_0^{(n+3)}/(g'q_\alpha))^{1/(\alpha+2)}$.

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